§3. Topological Stabilizer Codes

§3.1 $\mathbb{Z}_2$ Chain Complex

Consider a surface $G = (V, E, F)$ that consists of the vertices $V = \{v_k\}$, edges $E = \{e_\ell\}$, and faces $F = \{f_m\}$.
(G is embedded on a 2d manifold, the "surface", on which the faces are defined)

Define a vector space C_0 over $\mathbb{Z}_2$, using each vertex $v_k \in V$ as a basis $\mathcal{B}(C_0) = \{v_k\}$, write for $c_0 \in C_0$

$$c_0 = \sum_k z_k v_k$$

where $z_k \in \{0, 1\}$

$\longrightarrow$ C_0 is abelian group under component wise addition (mod 2)

Similarly, define abelian groups C_1 and C_2 using edges $\{e_\ell\}$ and faces $\{f_m\}$, i.e $\mathcal{B}(C_1) = \{e_\ell\}$, $\mathcal{B}(C_2) = \{f_m\}$

$$\rightarrow \quad c_1 = \sum_\ell z_\ell e_\ell \qquad 1\text{-chain}$$

$$c_2 = \sum_m z_m f_m \qquad 2\text{-chain}$$

$$z_\ell, z_m \in \{0, 1\}$$

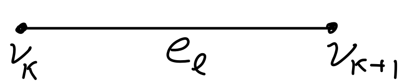
Next, define a homomorphism

$$\partial_i : C_i \longrightarrow C_{i-1}$$

such that

$$\partial_i \circ \partial_{i-1} = 0$$

Specifically, $\partial_i c_i$ is $(i-1)$-chain which is the boundary of c_i.

example: 

$\Rightarrow \partial e_\ell = v_k + v_{k+1}$

if $\partial c_i = 0 \longrightarrow c_i$ is called "cycle"

an i-cycle c_i is trivial if

$\exists$ $(i+1)$-chain c_{i+1} s.th. $c_i = \partial c_{i+1}$

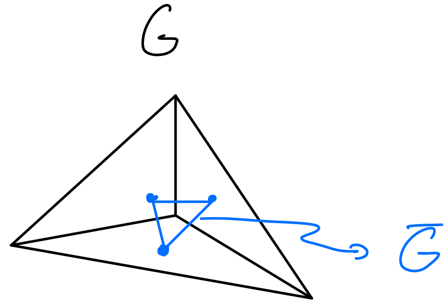
The homology group H_i is defined by

$$H_i = \ker(\partial_i)/\mathrm{Img}(\partial_{i+1})$$

$h \in H_i$ is called homology class

if $c_i \sim c_i' \sim h \longrightarrow c_i = c_i' + \partial c_{i+1}$

We also define the dual surface $\overline{G} = (\overline{V}, \overline{E}, \overline{F})$ with $\overline{V} = F$, $\overline{E} = E$, $\overline{F} = V$



Define $\mathbb{Z}_2$ chain complex on $\overline{G}$ using dual bases $\overline{B}(C_i)$, a dual i-chain $\overline{c}_i \in \overline{C}_i$ and a boundary operator

$$\overline{\partial}_i : \overline{C}_i \longrightarrow \overline{C}_{i-1}$$

<u>Surface code:</u>

- a qubit is defined on each edge $e_e \in E$ of surface G

- the Pauli product is defined as

$$W(c_1) = \prod_e W_e^{z_e} \quad \text{for 1-chain } c_1$$

where $W_e \in \{X_e, Y_e, Z_e\}$

We have $W(c_1)\,W(c_1') = W(c_1 + c_1')$

Consider two operators $X(c_1)$ and $Z(c_1')$

then define $c_1 \cdot c_1' \equiv \sum_e z_e z_e' \mod 2$

$\longrightarrow c_1 \cdot c_1' = 0$, iff $X(c_1)\,Z(c_1') = Z(c_1')\,X(c_1)$

$c_1 \cdot c_1' = 1$, iff $X(c_1)\,Z(c_1') = -Z(c_1')\,X(c_1)$

Let $M(\partial_i)$ be a matrix rep. of ∂_i with respect to the basis vectors $B(C_i)$ and $B(C_{i-1})$ $\longrightarrow$ $\left(M(\partial_i)\,c_i\right) \cdot c_{i-1} = c_i \cdot \left(M(\partial_i)^T c_{i-1}\right)$

Moreover, $M(\partial_1) = M(\bar{\partial}_2)^T$, $M(\partial_2) = M(\bar{\partial}_1)^T$

$$\partial_1 \circ \partial_2 = 0 \longrightarrow M(\bar{\partial}_2)^T M(\partial_2) = 0$$

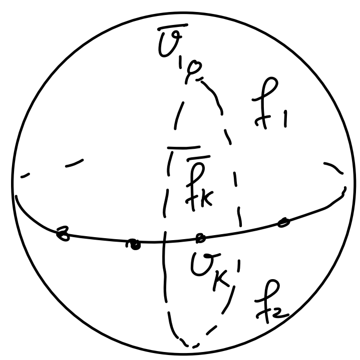
and so $\bar{\partial}\bar{c}_2 \cdot \partial c_2 = 0$, i.e.

$X(\partial c_2)$ and $Z(\bar{\partial}\bar{c}_2)$ commute

§3.2 A Bit-Flip Code: Exercise

Consider a regular polygon $G(V, E, F)$ on a sphere consisting of $n = |E|$ edges and two faces $F = \{f_1, f_2\}$

$\longrightarrow$ # qubits $= n$



We define a stabilizer generator for each dual face $\overline{f}_k = v_k$ as follows:

$$A_k = Z(\partial \overline{f}_k) = \prod_{e \in \delta v_k} Z_e$$

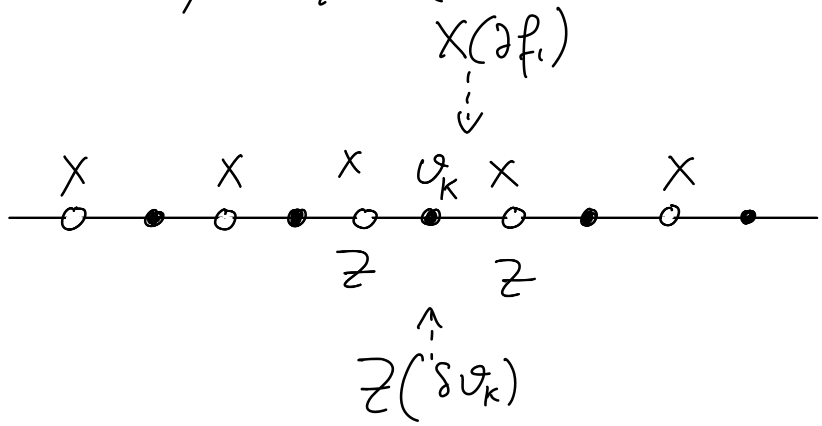
δv_k: set of edges incident to vertex v_k

Because $\prod_{v_k \in V} A_k = I$, there are $n-1$ indep. stabilizer generators.

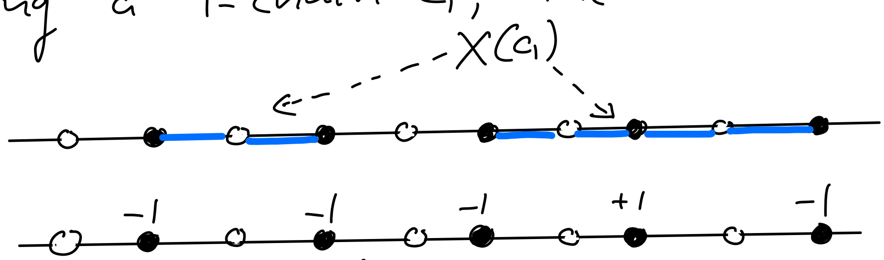
Define logical X operator

$$L_X = X(\partial f_1) = X(c_1)$$

Note that $L_X = X(\partial f_1)$ and $A_k = Z(\partial f_k)$ commute, $L_Z = Z_e$



A string of bit errors $X(c_1)$ is defined using a 1-chain c_1, the "error chain":



error syndrome: ∂c_1

error is detected by measuring the eigenvalues of the eigenvalues of the stabilizer generators $A_k = Z(\delta v_k)$

we have $c_1 \cdot \delta v_k = \partial c_1 \cdot v_k$

$\rightarrow X(c_1)$ anticommutes with A_k for $\partial c_1 \cdot v_k = 1$

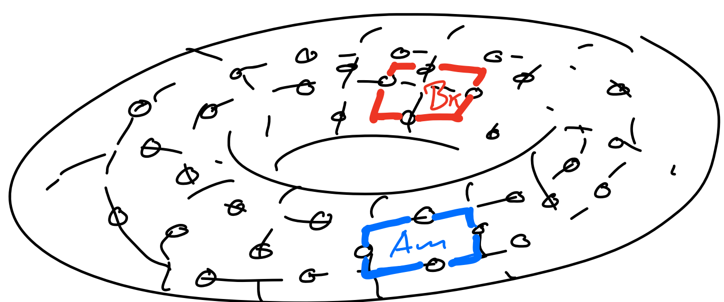
$\rightarrow \; v_K \in \partial c_1 \;\rightarrow\;$ eigenvalue of A_K becomes -1

error correction: find c_1' such that $\partial(c_1 + c_1') = 0$

§ 3.3 Definition of Surface Codes

1) Surface Code on a torus: Toric Code

Consider a square lattice $G = (V, E, F)$ on a torus with periodic b.c.'s



A_m: plaquette op

B_K: star op.

together with dual square lattice $\overline{G}$ define stabilizer generators for each face f_m and vertex v_K:

$$A_m = Z(\partial f_m)\,, \quad B_K = X(\delta v_K) = X(\partial \overline{f}_K)$$

As $\partial f_m \cdot \partial \overline{f}_K = 0 \;\longrightarrow\; [A_m, B_K] = 0$

The code state $|\psi\rangle$ satisfies

$A_m |\psi\rangle = |\psi\rangle$, $B_k |\psi\rangle = |\psi\rangle$ $\forall f_m \in F, v_k \in V$

For homologically equivalent 1-chains c_1 and c_1' we have

$$Z(c_1') = Z(c_1)\, Z(\partial c_2) = Z(c_1) \Big(\prod_{f_m \in c_2} A_m \Big)$$

$\longrightarrow$ actions on code space are same!

we write $Z(c_1') \sim Z(c_1)$